12/3

Stokes's Theorem

If S is "nice" surface with a "really nice" boundary and $\vec{F}$ is a v.f. on $\mathbb{R}^3$ w/ components having cts. potenial derivatives on S, then

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S}$$

NB: ① $\operatorname{curl}(\vec{F})$ is sometimes "nicer" than $\vec{F}$ so the computation is simpler...

② Sometimes the line integral is easier than the surface integral

Ex: Compute the ~~integ~~ $\iint_S \operatorname{curl}(\vec{F}) d\vec{S}$ for $\vec{F} = \langle xz, yz, xy \rangle$ and S the part of the sphere $x^2+y^2+z^2 = 4$ inside the cylinder $x^2+y^2 = 1$ and above xy-plane

Sol ① (compute the integral directly):

First we'll parameterize S: via $\vec{s}(r,\theta) = \langle r\cos\theta, r\sin\theta, \sqrt{4-r^2} \rangle$

on $(r,\theta) = [0,1] \times [0,2\pi]$

$$\vec{S}_r = \langle \cos\theta, \sin\theta, \tfrac{1}{2}(4-r^2)^{-\frac{1}{2}}(-2r) \rangle =$$
$$\langle \cos\theta, \sin\theta, -r(4-r^2)^{-\frac{1}{2}}$$

$$\vec{S}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

$$\vec{S}_r \times \vec{S}_\theta = \det \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & -r(4-r^2)^{-\frac{1}{2}} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \langle r^2(4-r^2)^{-\frac{1}{2}}\cos\theta, +(r^2(4-r^2)^{-\frac{1}{2}}, r\cos^2\theta + r\sin^2\theta \rangle$$

$$= \langle r^2(4-r^2)^{-\frac{1}{2}}\cos\theta, r^2(4-r^2)^{-\frac{1}{2}}, r \rangle$$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \det \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ xz & yz & xy \end{vmatrix}$$

$$= \langle x-y, -(y-x), 0 \rangle = (x-y)\langle 1,1,0 \rangle$$

$$\therefore \text{Curl}(\vec{F})(\vec{S}(r,\theta)) = (r\cos\theta - r\sin\theta)\langle 1,1,0 \rangle$$

$\therefore \text{curl}(\vec{F})(\vec{S}(r,\theta)) \cdot (\vec{S}_r \times \vec{S}_\theta)$

$= r(\cos\theta - \sin\theta)\left(r^2(4-r^2)^{-1/2}\cos\theta + r^2(4-r^2)^{-\frac{1}{2}}\sin\theta\right.$

$= r \cdot r^2 (4-r^2)^{-\frac{1}{2}}(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$

$= r \cdot r^2 (4-r^2)^{-\frac{1}{2}}(\cos^2\theta - \sin^2\theta)$

$= r \cdot r^2 (4-r^2)^{-\frac{1}{2}}(\cos 2\theta)$

$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \iint_D \text{curl}(\vec{F})(\vec{S}(r,\theta)) \cdot (\vec{S}_r \times \vec{S}_\theta)\, dA$

$= \iint_D r \cdot r^2 (4-r^2)^{-\frac{1}{2}} \cos 2\theta \, dA$

$= \int_{\theta=0}^{2\pi} \cos 2\theta \int_{r=0}^{1} r \cdot r^2 (4-r^2)^{-\frac{1}{2}}\, dr\, d\theta$

$u = 4 - r^2 = r^2 = 4 - u \qquad = \int_0^{2\pi} \cos 2\theta \int_{u=4}^{3} (4-u)u^{-\frac{1}{2}}\, du\, d\theta$

$du = -2r\,dr \qquad = \frac{1}{2}\int_0^{2\pi} \cos 2\theta \int_3^4 \left(4u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right) du\, d\theta$

$u(1) = 3 \quad u(0) = 4$

$$= \frac{1}{2}\int_0^{2\pi} \cos 2\theta \left[8u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}} \right]_3^4 d\theta$$

$$= \left(\left(8 - \frac{8}{3}\right) - \left(4\sqrt{3} - \sqrt{3}\right)\right) \int_0^{2\pi} \cos 2\theta \, d\theta$$

$$= \left(8 - \frac{8}{3} - 3\sqrt{3}\right)\left[\frac{1}{2}\sin(2\theta)\right]_0^{2\pi}$$

$$= \left(8 - \frac{8}{3} - 3\sqrt{3}\right)(0) = 0$$

Sol ② (Using Stokes's Theorem)

- We parameterize the boundary ∂S via

$$\vec{r}(\theta) = \langle \cos(\theta), \sin(\theta), \sqrt{3} \rangle$$

for $\theta = [0, 2\pi]$

$$\vec{r}\,'(\theta) = \langle -\sin\theta, \cos\theta, 0 \rangle$$

and

$$\vec{F}(\vec{r}(\theta)) = \langle \sqrt{3}\cos\theta, \sqrt{3}\sin\theta, \sin\theta\cos\theta \rangle$$

so

$$\vec{F}(\vec{r}(\theta)) \cdot (r'(\theta)) =$$

$$-\sin\theta\sqrt{3}\cos\theta + \cos\theta\sqrt{3}\sin\theta + 0 \rangle$$

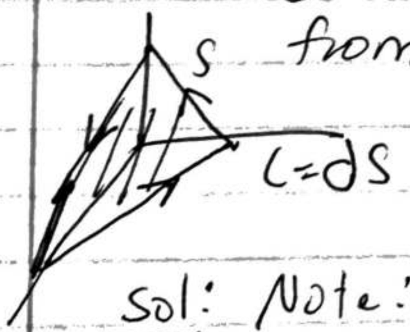
$$= 0$$

$$\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{s} \underset{\text{by stokes's}}{=} \int_{\partial S} \vec{F} \cdot d\vec{r} = \int_{\theta=0}^{2\pi} F(\vec{r}(\theta)) \cdot r'(\theta) d\theta = 0$$

③ If S and T are surfaces with $\partial S = \partial T$, then $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C F \cdot dr = \iint_T \operatorname{curl}(\vec{F}) \cdot d\vec{S}$

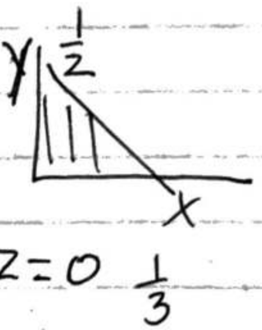
when $S \cup T$ does not enclose a point of discts. of curl$(\vec{F})$

Ex: Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = \langle 1, x+yz, xy-\sqrt{z} \rangle$ on C the intersection of plane $3x+2y+z=1$ with the coordinate planes in the first octant, orientated ~~clockw~~ counterclockwise from above



Sol: Note: This curve has three "pieces"

To parameterize S:



$S(x,y) = \langle x, y, 1-3x-2y \rangle$

$D = \{(x,y) : 0 \le x \le \frac{1}{3},\ 0 \le y \le -\frac{3}{2}x + \frac{1}{2}\}$

$\therefore \cancel{\vec{F}(S(x,y)) = \langle 1, x+y(1-3x-2y), xy - \sqrt{1-3x-2y} \rangle}$

$\mathrm{curl}(\vec{F}) = \nabla \times \vec{F} =$

$$\det \begin{bmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ 1 & x+yz & \cancel{xy}\ \sqrt{z} \end{bmatrix}$$

$= \langle x-y, -(y-0), 1-0 \rangle = \langle x-y, -y, 1 \rangle$

$\mathrm{curl}(\vec{F})(\vec{S}(x,y)) = \cancel{\langle x-y, -y, 1 \rangle}$

$= \langle x-y, -y, 1 \rangle$

$S_x = \langle 1, 0, -3 \rangle$

$S_y = \langle 0, 1, -2 \rangle$

$$\vec{S}_x \times \vec{S}_y = \det \begin{vmatrix} i & j & k \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{vmatrix} =$$

$\langle 3, -(-2), 1 \rangle = \langle 3, 2, 1 \rangle$

$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \mathrm{curl}(\vec{F})\, d\vec{S}$

Stokes

$= \iint_D \mathrm{curl}(\vec{F})(S(x,y)) \cdot (S_x \times S_y)\, dA$

$=$

$$\int_{x=0}^{\frac{1}{3}} \int_{x=0}^{-\frac{3}{2}x+\frac{1}{2}} (3x-3y-2y+1)\,dy\,dx$$

$$=\int_{x=0}^{\frac{1}{3}} \int_{y=0}^{-\frac{3}{2}x+\frac{1}{2}} (3x-5y+1)\,dy\,dx$$

$$\int_{x=0}^{\frac{1}{3}} \left[3xy-\frac{5}{2}y^2+y\right]_0^{-\frac{3}{2}x+\frac{1}{2}} dx$$

$$\int_{x=0}^{\frac{1}{3}} 3x\left(-\frac{3}{2}x+\frac{1}{2}\right)-\frac{5}{2}\left(-\frac{3}{2}x+\frac{1}{2}\right)^2+\left(-\frac{3}{2}x+\frac{1}{2}\right) dx$$

$$\int_0^{\frac{1}{3}} \left(-\frac{9}{2}x^2+\frac{3}{2}x-\frac{5}{2}\left(\frac{9}{4}x^2+\frac{-3}{2}x+\frac{1}{4}\right)-\frac{3}{2}x+\frac{1}{2}\right) dx$$

$$\int_0^{\frac{1}{3}} -\frac{81}{8}x^2+\frac{15}{4}x-\frac{1}{8}) dx$$

$$=\frac{1}{8}\left[-\frac{81}{3}x^3+15x^2-x\right]_0^{\frac{1}{3}}$$

$$=\frac{1}{8}\left(-\frac{81}{3\cdot 3^3}+\frac{15}{3^2}-\frac{1}{3}-0\right)$$

$$=\frac{1}{8}\left(-1+\frac{5}{3}-\frac{1}{3}\right)=\frac{1}{8}\cdot\frac{1}{3}=\boxed{\frac{1}{24}}$$

Exercise: Compute $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F} = \langle 2y, xz, x+y \rangle$ and C the curve of intersection of the plane $z = y+2$ and the cylinder $x^2+y^2 = 4$